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Short Communication

A study on orthogonality in generalized normed spaces

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Abstract

In this paper, Cauchy-Schwarz inequality on n-inner product spaces is reproved, and notions of orthogonality on n-normed spaces are introduced. This is the first approach to orthogonality types in such spaces.

1. Introduction

Functional analysis as an independent mathematical discipline started at the turn of the 19th century and was finally established in the 1920s and 1930s, on the one hand under the influence of the study of specific classes of linear operators integral operators and integral equations connected with them and on the other hand under the influence of the purely intrinsic development of modern mathematics with its desire to generalize and thus to clarify the true nature of some regular behavior. Quantum mechanics also had a great influence on the development of functional analysis, since its basic concepts, for example, energy, turned out to be linear operators (which physicists at first rather loosely interpreted as infinite-dimensional matrices) on infinite-dimensional spaces. Recent studies of the sections of functional analysis we refer [1-13].

Orthogonality is one of the branches of functional analysis. Many authors have developed several notions of orthogonality in a normed space. For example, the following definitions of Pythagorean, Isosceles, and the Birkhoff-James orthogonality in a real normed space $(X, \|\cdot\|)$ are known:

P- Orthogonality: x is P- orthogonality to y (denoted by $x \perp_{\mathrm{P}} y$) if only if:

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2. \tag{1.1}$$

I-Orthogonality: x is I- orthogonality to y (denoted by $x \perp_{\mathrm{I}} y$) if only if:

$$\|x+y\| = \|x-y\|. \tag{1.2}$$

BJ-Orthogonality: x is BJ- orthogonality to y (denoted by $x \perp_{\mathrm{BJ}} y$) if only if:

$\|x+\alpha y\| \geq \|x\|$ for every $\alpha \in \mathbb{R}$.

2. Preliminary

We recall some preliminary definitions related to our findings as presented here below:

Definition 2.1. (n–normed spaces) [14] Let X be a real vector space of dim $\geq n$. An n–norm on X is a mapping $\|\cdot,\dots,\cdot\| : X^n \to \mathbb{R}$, which satisfies the following four conditions:

nN 1: $\|x_1,\dots,x_n\| = 0$, if and only if $x_1,\dots,x_n$ are linearly dependent,

n N 2: $\|x_1,\dots,x_n\| = \|x_{i_1},\dots,x_{i_n}\|$, for every permutation $(i_1,\dots i_n)$ of $(1,\dots,n)$,

nN 3: $\|\alpha x_1,\dots,x_n\| = |\alpha|\|x_1,\dots,x_n\|$ for $\alpha \in \mathbb{R}$,

n N 4: $\|x_1 + \acute{x}_1, x_2,\dots,x_n\| \leq \|x_1,x_2,\dots,x_n\| + \|\acute{x}_1,x_2,\dots,x_n\|$, for all $x_1,\acute{x}_1,x_2,\dots,x_n \in X$. The pair $(X,\|\cdot,\dots,\cdot\|)$ is called an n–normed spaces.

Definition 2.2. (n–inner product spaces) [5] A real-valued function $\langle\cdot,\cdot \mid \cdot,\dots,\cdot\rangle$ on X^{n+1} satisfied the following properties:

Definition 3.1. Suppose X is an n–inner product spaces, for $x,y,x_2,\dots,x_n \in X$, we say that $x,x_2\dots,x_n$ is orthogonal to $y,,x_2,\dots,x_n$ if $\langle x,y|x_2\dots,x_n\rangle = 0$.

Note that,

$\langle x,y|x_2\dots,x_n\rangle = 0$ if and only if $\langle y,x|x_2\dots,x_n\rangle = 0$, by nI 3. So, x is orthogonal to y if and only if y is orthogonal to x, so we often say simply that x and y are orthogonal.

As an illustration of its use, let`s generalize Pythagoras` relation (1.1) and Isosceles relation (1.2) by using n–inner product spaces.

Theorem 3.1. Suppose X is an n–inner product spaces $(X,\langle\cdot,\cdot \mid \cdot,\dots,\cdot\rangle)$ and $x,y \in X$ are orthogonal, then

$\|x+y,x_2,\dots,x_n\|^2 = \|x,x_2\dots,x_n\|^2 + \|y,x_2,\dots,x_n\|^2 = \|x-y,x_2,\dots,x_n\|^2$,

for every $x_2,\dots,x_n \in X$.

Proof. Since $x-y,x_2,\dots,x_n = x+(-y),x_2,\dots,x_n$, the statement about $x-y,x_2,\dots,x_n$, follows from the statement for $x+y,x_2,\dots,x_n$, and by using nN 3 we get $\|-y,x_2,\dots,x_n\| = \|y,x_2,\dots,x_n\|$.

Now, by using equation nI 5,

$\langle x+y,x+y|x_2,\dots,x_n\rangle = \langle x,x+y|x_2,\dots,x_n\rangle + \langle y,x+y|x_2,\dots,x_n\rangle$

$= \langle x,x|x_2,\dots,x_n\rangle + \langle x,y|x_2,\dots,x_n\rangle + \langle y,x,|x_2,\dots,x_n\rangle + \langle y,y|x_2,\dots,x_n\rangle$

$= \langle x,x|x_2,\dots,x_n\rangle + \langle y,y|x_2,\dots,x_n\rangle$.

By orthogonality of $x,x_2\dots,x_n$, $y,x_2,\dots,x_n$, and by the relation $\|x_1,x_2,\dots,x_n\|^2 = \langle x_1,x_1|x_2,\dots,x_n\rangle$, which mentioned in [6], proving the result.

nI 1: $\langle x_1,x_1|x_2,\dots,x_n\rangle \geq 0$ and $\langle x_1,x_1|x_2,\dots,x_n\rangle = 0$, if and only if $x_1,x_2\dots,x_n$ are linearly dependent.

nI 2: $\langle x_1,x_1|x_2,\dots,x_n\rangle = \langle x_{i_1},x_{i_1}|x_{i_2},\dots,x_{i_n}\rangle$, for any permutation $(i_1,\dots,i_n)$ of $(1,\dots,n)$.

nI 3: $\langle \acute{x}_1,x_1|x_2,\dots,x_n\rangle = \langle x_1,\acute{x}_1|x_2,\dots,x_n\rangle$,

nI 4: $\langle \alpha x_1,x_1|x_2,\dots,x_n\rangle = \alpha\langle x_1,x_1|x_2,\dots,x_n\rangle$, for every $\alpha \in \mathbb{R}$.

n I 5: $\langle x_0+\acute{x}_0,x_1|x_2,\dots,x_n\rangle = \langle x_0,x_1|x_2,\dots,x_n\rangle + \langle \acute{x}_0,x_1|x_2,\dots,x_n\rangle$.

is called an n–inner product on a vector spaces X. The pair $(X,\langle\cdot,\cdot \mid \cdot,\dots,\cdot\rangle)$ is called an n–inner product spaces.

Definition 2.3. (Cauchy-Schwarz inequality) [10,15] If $\langle x,y|x_2,\dots,x_n\rangle$ is a n–inner product on X, then we have:

$\langle x,y|x_2,\dots,x_n\rangle^2 \leq \langle x,x|x_2,\dots,x_n\rangle\langle y,y|x_2,\dots,x_n\rangle$,

3. Results and discussion

Also, we add one use of orthogonality on an n–inner product spaces $(X,\langle\cdot,\cdot \mid \cdot,\dots,\cdot\rangle)$ as the following:

Theorem 3.2. Suppose $x,y \in X$, $y \neq 0$, where X an n–inner product spaces then there exist a unique $x_1, \grave{x}_1 \in X$ such that

$x = x_1 + \grave{x}_1$, $x_1 = cy$, for some $c \in \mathbb{C}$, and $\langle \grave{x}_1,y|x_2,\dots,x_n\rangle = 0$, (3.1)

for all $x_2,\dots,x_n \in X$.

Proof. If $x = x_1 + \grave{x}_1$ then taking the n–inner product spaces $(X,\langle\cdot,\cdot \mid \cdot,\dots,\cdot\rangle)$ with y and using $x_1 = cy$ we deduce:

$\langle x,y|x_2,\dots,x_n\rangle = \langle x_1+\grave{x}_1,y|x_2,\dots,x_n\rangle$

$= \langle x_1,y|x_2,\dots,x_n\rangle + \langle \grave{x}_1,y|x_2,\dots,x_n\rangle$, by using nI 5

$= \langle cy,y|x_2,\dots,x_n\rangle + \langle \grave{x}_1,y|x_2,\dots,x_n\rangle$

$= c\langle y,y|x_2,\dots,x_n\rangle + 0$, by equation (3.1)

$= c\|y,x_2,\dots,x_n\|^2$,

so as $y \neq 0$, $c = \frac{\langle x,y|x_2,\dots,x_n\rangle}{\|y,x_2,\dots,x_n\|^2}$.

Thus, $\langle x_1,y|x_2,\dots,x_n\rangle = \langle cy,y|x_2,\dots,x_n\rangle$ and $\langle \grave{x}_1,y|x_2,\dots,x_n\rangle = \langle x-cy,y|x_2,\dots,x_n\rangle$, given a uniqueness.

On the other hand, if we let $c = \frac{\langle x,y|x_2,\dots,x_n\rangle}{\|y,x_2,\dots,x_n\|^2}$, $\langle x_1,y|x_2,\dots,x_n\rangle = \langle cy,y|x_2,\dots,x_n\rangle$

and $\langle \grave{x}_1,y|x_2,\dots,x_n\rangle = \langle x-cy,y|x_2,\dots,x_n\rangle$.

Then $x = x_1 + \grave{x}_1$ and $x_1 = cy$ are satisfied, so we merely need to check $\langle \grave{x}_1,y|x_2,\dots,x_n\rangle = 0$. But

$\langle \grave{x}_1, y|x_2, \dots, x_n\rangle = \langle x - cy, y|x_2, \dots, x_n\rangle$
$= \langle x, y|x_2, \dots, x_n\rangle + \langle -cy, y|x_2, \dots, x_n\rangle$
$= \langle x, y|x_2, \dots, x_n\rangle - c\langle y, y|x_2, \dots, x_n\rangle$,
by using nI 4
$= \langle x, y|x_2, \dots, x_n\rangle - \frac{\langle x, y|x_2, \dots, x_n\rangle}{\|y, x_2, \dots, x_n\|^2}$
$\|y, x_2, \dots, x_n\|^2 = 0.$
So, the desired vectors x_1 and $\grave{x}_1$ indeed exist.
In order to make this a useful tool, we need to be able to estimate the n−inner product $\langle \cdot, \cdot | \cdot, \dots, \cdot\rangle$ using n-norm $\|\cdot, \dots, \cdot\|$. It is worthy to note that the above results will be used to achieve and reprove by the Cauchy-Schwarz\` inequality which mentioned in relation (2.1), as following:

Lemma 3.1. In an $n-$ inner product spaces $(X, \langle \cdot, \cdot \mid \cdot, \dots, \cdot\rangle)$
$|\langle x, y|x_2, \dots, x_n\rangle| \leq \|x, x_2 \dots, x_n\| \|y, x_2, \dots, x_n\|$ for all $x, y, x_2 \dots, x_n \in X$.

Proof. If $x = 0$, then the both sides vanish, so we may assume $x \neq 0$.
Write:
$x = x_1 + \grave{x}_1$ as in **Theorem 3.2**, so, $x_1 = cy$, $c = \frac{\langle x, y|x_2, \dots, x_n\rangle}{\|y, x_2, \dots, x_n\|^2}$.
Then by using
$\langle x_1, \grave{x}_1|x_2 \dots, x_n\rangle = c\langle y, \grave{x}_1|x_2 \dots, x_n\rangle = 0,$
$\|x, x_2 \dots, x_n\|^2 = \|x_1, x_2 \dots, x_n\|^2 + \|\grave{x}_1, x_2 \dots, x_n\|^2,$
$\geq \|x_1, x_2 \dots, x_n\|^2$
$= |c|^2 \|y, x_2 \dots, x_n\|^2$
$= \frac{|\langle x, y|x_2, \dots, x_n\rangle|^2}{\|y, x_2 \dots x_n\|^2}.$
multiplying through by $\|y, x_2 \dots, x_n\|^2$ and taking the non-negative square root completes the proof of the Lemma.

Conclusions

In this work, Cauchy-Schwarz inequality on n-inner product spaces is reproved, and notions of orthogonality on n-normed spaces are introduced. This is the first approach to orthogonality types in such spaces. The problem considered in this paper can be generalized to a higher dimension involving a general formulation of orthogonality relation in real normed linear spaces via norm derivatives.

Authors' contributions

Conceptualization, B.H. and A.H.; methodology, B.H. and A.A.; validation, B.H., A.H. and J.P.; formal analysis, B.H ;.resources, B.H. and A.A.; data curation, A.H.; writing-original draft preparation, B.H.; writing-review and editing, B.H. and A.A.; supervision, J.P. and K.G. All authors have read and agreed to the published version of the manuscript.

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Conflicts of interest

The authors declare that they have no competing interests regarding this research work.

References

1. Alonso, J. Uniqueness Properties of Isosceles Orthogonality in Normed Linear Spaces. *Ann. Sci. Math. Qu. ebec.* 1994, 18(1), 25-38.
2. Ahire, Y.M.; Patil1, J.; Hardan, B.; Hamoud, A.A.; Bachhav, A. Recent Advances on Fixed Point Theorems, *Bull. Pure Appl. Sci. Sect. E Math. Stat.* **2022,** 41(1), 1-11.
3. Patil, J.; Hardan, B.; Hamoud, A.; Bachhav, A. ; Emadifar, H. A new result on Branciari metric space using (α; γ)-contractive mappings. *Topol. Algebra Appl.* 2022, 10(1), 103-112.
4. Cho, Y.J.; Kim. S. S. Gateaux Derivatives and 2-Inner Product Spaces. *Glas. Mat. Ser. III.* 1983, (47), 197-203.
5. Gunawan, H. On n-Inner Products, n-Norms and the Cauchy Schwarz Inequality. *Sci. Math. JPN.* **2002**, 55, 53-60.
6. Hamoud, A. A.; Patil, J.; Hardan, B.; Bachhav, A; Emadifar, H.; Guunerhan, H. Generalizing contractive mappings on b-rectangular metric space. *Adv. Math. Phys.* 2022, 1-10.
7. Hardan, B.; Patil, J.; Chaudhari A.; Bachhav, A. Approximate fixed points for n-Linear functional by-nonexpansive Mappings on -Banach spaces. *J. Mat. Anal. Model.* **2020**, 1(1), 20-32.
8. Hardan, B.; Patil, J.; Chaudhari A.; Bachhav, A. Caristi Type Fixed Point Theorems ofContractive Mapping with Application. One Day National Conference on Recent AdvancesIn Sciences Held on: 13th February 2020. 609-614.
9. Khan, A.; Siddiqui, A. B-orthogonality in 2-Normed Space, Bull. *Calcutta Math. Soc.* 1982, 74, 216-222.
10. Meyer, C.D. Matrix analysis and applied linear algebra, Ed.1, SIAM. 2004.
11. Patil, J.; Hardan, B.; Hamoud, A. A.; Bachhav, A.; Emadifar, H.; Ghanizadeh, A., Edalatpanah, S.A.; Azizi, H. On (η,γ)_(f,g)-Contractions in Extended b-Metric Spaces. *Adv. Math. Phys.* 2022, 1-8.
12. Patil, J.; Hardan, B. On Fixed Point Theorems in Complete Metric Space. *J. Comput. Math. Sci.* 2019, 10, 1419-1425.